June 2010 Core 4

① (a) $f\left(\frac{1}{4}\right) = 8\left(\frac{1}{4}\right)^3 + 6\left(\frac{1}{4}\right)^2 - 14\left(\frac{1}{4}\right) - 1$

$= \frac{8}{64} + \frac{6}{16} - \frac{14}{4} - 1$

$= \frac{1}{8} + \frac{3}{8} - \frac{28}{8} - \frac{8}{8} = \frac{-32}{8} = -4$

(b)(i) $g(x) = 8x^3 + 6x^2 - 14x + d$

$(4x-1)$ is a factor $\Rightarrow g\left(\frac{1}{4}\right) = 0$

$g\left(\frac{1}{4}\right) = \frac{-24}{8} + d$

$\Rightarrow -3 + d = 0$ $\qquad \underline{\underline{d = 3}}$

(ii) Comparing coefficients or:

$$\begin{array}{r} 2x^2 + 2x - 3 \\ 4x-1 \overline{\smash{)}\, 8x^3 + 6x^2 - 14x + 3} \\ -\;\; 8x^3 - 2x^2 \\ \hline 8x^2 - 14x + 3 \\ -\;\; 8x^2 - 2x \\ \hline -12x + 3 \\ -\;\; -12x + 3 \\ \hline 0 \end{array}$$

$a = 2 \quad b = 2 \quad c = -3$

② (a) $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$\frac{dy}{dt} = 6t \qquad \frac{dx}{dt} = -3 \Rightarrow \frac{dt}{dx} = \frac{-1}{3}$

$\frac{dy}{dx} = \frac{-6t}{3} = -2t$

(b) $\underline{t = 1}$ $\quad x = 1 - 3 = -2 \quad y = 1 + 2 = 3$

$m_{\text{(tangent)}} = -2 \qquad m_{\text{normal}} = \frac{1}{2}$

$\Rightarrow \underline{\underline{y - 3 = \frac{1}{2}(x + 2)}}$

c) $x = 1-3t$

$3t = 1-x$

$t = \dfrac{1-x}{3} \Rightarrow y = 1+2t^3$

$= 1+2\left(\dfrac{1-x}{3}\right)^3$

$y = 1+\dfrac{2}{27}(1-x)^3$

3) $$\frac{7x-3}{(x+1)(3x-2)} = \frac{A}{(x+1)} + \frac{B}{(3x-2)}$$

$$= \frac{A(3x-2)+B(x+1)}{(x+1)(3x-2)}$$

$7x-3 = A(3x-2)+B(x+1)$

$\underline{\underline{x=-1}}$ $\quad -10 = A(-5) \Rightarrow \underline{\underline{A=2}}$

$\underline{\underline{x=\frac{2}{3}}}$ $\quad 7\left(\frac{2}{3}\right)-3 = B\left(\frac{5}{3}\right)$

$\frac{14}{3}-\frac{9}{3} = B\left(\frac{5}{3}\right)$

$\frac{5}{3} = \frac{5}{3}B \Rightarrow \underline{\underline{B=1}}$

(ii) $$\int \frac{2}{(x+1)} + \frac{1}{(3x-2)} \, dx = 2\ln(x+1) + \frac{1}{3}\ln(3x-2) + C$$

b) $$P + \frac{Qx+R}{2x^2-x+1} = \frac{P(2x^2-x+1)}{(2x^2-x+1)} + \frac{Qx+R}{(2x^2-x+1)}$$

$6x^2+x+2 = P(2x^2-x+1) + Qx + R$

$= 2Px^2 + (Q-P)x + (P+R)$

equating coefficients: $6=2P \Rightarrow \underline{P=3}$ $\quad Q-P=1 \Rightarrow \underline{Q=4}$

$P+R=2 \Rightarrow \underline{R=-1}$ $\rightarrow$

$$3 + \frac{4x-1}{2x^2-x+1}$$

4 (a)(i) $(1+x)^{\frac{3}{2}} \approx 1 + \left(\frac{3}{2}\right)x + \frac{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)x^2}{2!}$

$$= 1 + \frac{3}{2}x + \frac{3}{8}x^2$$

(ii) $(16+9x)^{\frac{3}{2}} = \left(16\left(1+\frac{9}{16}x\right)\right)^{\frac{3}{2}} = 64\left(1+\frac{9}{16}x\right)^{\frac{3}{2}}$

now replace x with $\frac{9}{16}x$ in (a)(i) and $\times 64$.

$$64\left(1 + \frac{3}{2}\left(\frac{9}{16}x\right) + \frac{3}{8}\left(\frac{9}{16}x\right)^2\right)$$

$$= 64\left(1 + \frac{27}{32}x + \frac{243}{2048}x^2\right)$$

$$= 64 + 54x + \frac{243}{32}x^2$$

(b) let $16 + 9x = 13 \Rightarrow 9x = -3 \rightarrow x = \frac{-1}{3}$

$$13^{\frac{3}{2}} \approx 64 + 54\left(\frac{-1}{3}\right) + \frac{243}{32}\left(\frac{-1}{3}\right)^2$$

$$= 64 - 18 + \frac{27}{32}$$

$$= 46 + \frac{27}{32}$$

$a = 27$ $b = 32$

5 (a)(i) use double angle formula: $\cos 2x = \cos^2 x - \sin^2 x$

$$= 1 - 2\sin^2 x$$

$3\cos 2x + 2\sin x + 1 = 0$

$3(1-2\sin^2 x) + 2\sin x + 1 = 0$

$3 - 6\sin^2 x + 2\sin x + 1 = 0 \rightarrow 6\sin^2 x - 2\sin x - 4 = 0$

$3\sin^2 x - \sin x - 2 = 0$

(ii) let $X = \sin x$

$3X^2 - X - 2 = 0$

$(3X+2)(X-1) = 0$

$X = \sin x = \underline{\underline{1}}$ or $\underline{\underline{-\frac{2}{3}}}$

(b)(i) $R\cos(2x - \alpha) = R(\cos(2x)\cos\alpha + \sin(2x)\sin\alpha)$

$= (R\cos\alpha)\cos 2x + (R\sin\alpha)\sin 2x$.

$3\cos 2x + 2\sin 2x = (R\cos\alpha)\cos 2x + (R\sin\alpha)\sin 2x$

① $3 = R\cos\alpha$

② $2 = R\sin\alpha$

$\frac{②}{①}$: $\frac{2}{3} = \frac{\sin\alpha}{\cos\alpha} \Rightarrow \tan\alpha = \frac{2}{3}$ $\underline{\underline{\alpha = 33.7°}}$

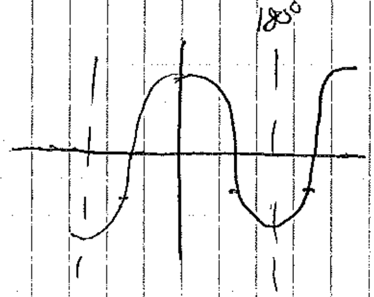
$①^2 + ②^2$: $3^2 + 2^2 = R^2\cos^2\alpha + R^2\sin^2\alpha$

$13 = R^2(\cos^2\alpha + \sin^2\alpha)$

$13 = R^2 \Rightarrow \underline{\underline{R = \sqrt{13}}}$

(ii) $\sqrt{13}\cos(2x - 33.7°) + 1 = 0$

$\cos(2x - 33.7) = \frac{-1}{\sqrt{13}}$



$2x - 33.7 = \pm 106.1°,$

$253.9°$

$\underline{\underline{x = 69.9°, 143.8°}}$

⑥ (a) $x^3y + \cos(\pi y) = 7$

$\underline{y=1}$: $x^3 + \cos\pi = 7$

$x^3 = 7 - \cos\pi = 8$

$\Rightarrow \underline{\underline{x=2}}$

product rule.

(b) $\frac{d}{dx}(x^3y) = \left(\frac{d}{dx}x^3\right)\times y + x^3\times\left(\frac{d}{dx}y\right)$

$= 3x^2y + x^3\left(\frac{dy}{dx}\frac{d}{dy}y\right)$

$= 3x^2y + x^3\frac{dy}{dx}$

$\frac{d}{dx}[\cos(\pi y)] = \frac{dy}{dx}\frac{d}{dy}(\cos(\pi y)) = -\pi\sin(\pi y)\frac{dy}{dx}$

$\Rightarrow \frac{d}{dx}(x^3y + \cos(\pi y)) = \frac{d}{dx}7$

$3x^2y + x^3\frac{dy}{dx} - \pi\sin(\pi y)\frac{dy}{dx} = 0$

$\frac{dy}{dx}\times\left[x^3 - \pi\sin(\pi y)\right] = -3x^2y$

$\frac{dy}{dx} = \frac{3x^2y}{\pi\sin(\pi y) - x^3}$

$\underline{y=1} \to \underline{x=2}$ (from (a))

$\frac{dy}{dx} = \frac{3(4)}{\pi\sin\pi - 8} = \frac{12}{-8} = \underline{\underline{-\frac{3}{2}}}$

7 (a) B: $\begin{pmatrix} -1+\mu \\ 3-2\mu \\ 4-\mu \end{pmatrix}$ $\mu = 2 \rightarrow \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} - \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \\ 0 \end{pmatrix}$

(b) (i) $r_1 = \begin{pmatrix} 4+2\lambda \\ -3 \\ 2+\lambda \end{pmatrix}$ $r_2 = \begin{pmatrix} -1+\mu \\ 3-2\mu \\ 4-\mu \end{pmatrix}$

1 $\quad 4+2\lambda = -1+\mu$

2 $\quad -3 = 3-2\mu$

3 $\quad 2+\lambda = 4-\mu$

Solve 1 and 2 simultaneously and then check in 3

2: $-6 = -2\mu \Rightarrow \underline{\underline{\mu = 3}}$

1: $4+2\lambda = -1+3$

$2\lambda = -2 \Rightarrow \underline{\underline{\lambda = -1}}$

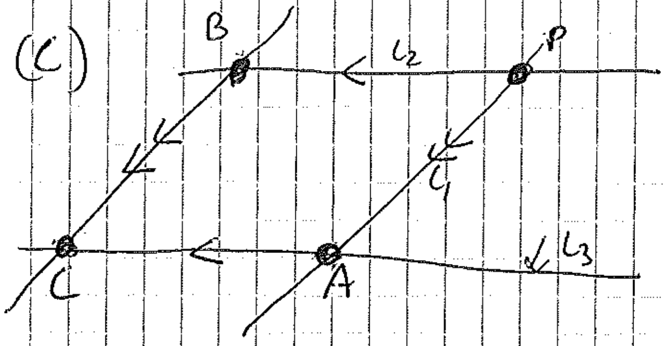
Check 3: $2+(-1) = 1$

$4-(3) = 1$ ✓ They do intersect.

(ii) Intersect when $\lambda = -1$: $\begin{pmatrix} 4-2 \\ -3 \\ 2-1 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$

P is $(2, -3, 1)$

(c)



$$r_3 = \begin{pmatrix}1\\-2\\2\end{pmatrix} + \gamma\begin{pmatrix}2\\0\\1\end{pmatrix}$$

B lies on line; parallel to l_1

$$\vec{OC} = \begin{pmatrix}1+2g\\-1\\2+g\end{pmatrix} \quad \text{for some value } g$$

$$|\vec{CB}| = |\vec{AP}|$$

$$\vec{AP} = \begin{pmatrix}2\\-3\\1\end{pmatrix} - \begin{pmatrix}4\\-3\\2\end{pmatrix} = \begin{pmatrix}-2\\0\\-1\end{pmatrix} \Rightarrow |AP| = \sqrt{(-2)^2+(-1)^2} = \sqrt{5}$$

$$\vec{CB} = \begin{pmatrix}1\\-1\\2\end{pmatrix} - \begin{pmatrix}1+2g\\-1\\2+g\end{pmatrix} = \begin{pmatrix}-2g\\0\\-g\end{pmatrix}$$

$$|\vec{CB}| = \sqrt{5} \Rightarrow \sqrt{(-2g)^2+(-g)^2} = \sqrt{5g^2} = \sqrt{5}$$

$$5g^2 = 5 \Rightarrow g^2 = 1$$

$$\underline{g = \pm 1}$$

$$C_1 = \begin{pmatrix}1+2(1)\\-1\\2+(1)\end{pmatrix} = \begin{pmatrix}3\\-1\\3\end{pmatrix}$$

$$C_2 = \begin{pmatrix}1+2(-1)\\-1\\2+(-1)\end{pmatrix} = \begin{pmatrix}-1\\-1\\1\end{pmatrix}$$

(8) (a) $\frac{dx}{dt} = \frac{-1}{5}(x+1)^{\frac{1}{2}}$ Separate variables

$\int (x+1)^{-\frac{1}{2}} dx = \int \frac{-1}{5} dt$

$2(x+1)^{\frac{1}{2}} = \frac{-1}{5}t + C$

$(x+1)^{\frac{1}{2}} = \frac{-1}{10}t + k \Rightarrow x+1 = \left(\frac{-1}{10}t + k\right)^2$

$x = \left(\frac{-1}{10}t + k\right)^2 - 1$

$\underline{t=0}$ $x=\underline{80}$ $\rightarrow$ $80 = k^2 - 1$

$\Rightarrow k = 9$

$x = \left(9 - \frac{1}{10}t\right)^2 - 1$

(b) $\underline{t = 60}$

$x = \left(9 - \frac{1}{10}(60)\right)^2 - 1 = 3^2 - 1 = 8\%$

(c)(i) $\frac{dA}{dt} \propto A(9-A) \Rightarrow \frac{dA}{dt} = K \times A(9-A)$ for constant K

(ii) $4.5 = \frac{9}{1+4e^{-0.09t}} \Rightarrow 4.5 + 18e^{-0.09t} = 9$

$\nearrow$ 50% of 9

$18e^{-0.09t} = 4.5$

$e^{-0.09t} = \frac{4.5}{18} = \frac{1}{4}$

$\ln\left(e^{-0.09t}\right) = \ln\left(\frac{1}{4}\right)$

$-0.09t = \ln\left(\frac{1}{4}\right)$

$t = \frac{-\ln\left(\frac{1}{4}\right)}{0.09} = \frac{\ln 4}{0.09} = \frac{100}{9}\ln 4 = 15.4$ hours.